ELSEVIER

Discrete Mathematics 177 (1997) 249–257

DISCRETE MATHEMATICS

Note

On blocking sets in a design

Luigia Berardi

Dipartimento di Ingegneria Elettrica, Università degli Studi de L'Aquila, I-67040 Monteluco di Roio L'Aquila, Italy

Received 2 December 1996; accepted 16 December 1996

Abstract

In this paper we give a bound for the cardinality of an intersection set of a 2-(v,k,λ) design D. We give a new proof of Drake's inequality for the cardinality of a blocking set of D. Our proof will enable us to characterize the case of equality. We investigate the existence of the blocking sets of type $(1,s)$ in a design. We prove some non-existence theorems and give some bounds for the parameters of a design containing such a blocking set. © 1997 Elsevier Science B.V.

1. Introduction

A 2-(v,k,λ) *design* D $(2<k<v,\ \lambda>0)$ is a pair (P,B), where P is a v-set of *points* and B a family of k-subsets of P, called *blocks*, such that through any two points there are exactly λ blocks.

It is possible that distinct blocks could be incident with the same set of points. In this case we say that the design has repeated blocks. A design without repeated blocks is distinguished by calling it a *simple* design. In this paper the designs we treat are always simple and we omit this qualifying adjective. A design is called *complete* if every set of k points is incident with a block.

A 2-(v,k,λ) design with $\lambda=1$ is called a Steiner system $S(2,k,v)$.

The number b of blocks and the number r of blocks through a point of a 2-(v,k,λ) design are given, respectively by

$$b=\frac{\lambda v(v-1)}{k(k-1)},\qquad r=\frac{\lambda(v-1)}{k-1}.$$

A 2-(v,k,λ) design with equally many points and blocks is called *symmetric*. It is easy to see that each point of a symmetric design is on exactly k blocks; moreover, we have $b=v=k(k-1)/\lambda+1$.

PII S0012-365X(96)00020-4

The integer $r-\lambda$ is called the *order* of the 2-(v,k,λ) design.

An *intersection s-set* in a 2-(v,k,λ) design D is a set S of s points of P such that each block intersects S. A *blocking s-set* in a 2-(v,k,λ) design D is an intersection s-set containing no block of D. So, also the complement point set $P-S$ of a blocking s-set S is a blocking set of D. A blocking s-set S of D is called *irreducible* (or *minimal*) if the set $S-\{x\}$ is not a blocking set for all $x\in S$. We say that D is an *s-blocked design*, if s is the minimum cardinality of a blocking set in D.

The *index* of a blocking s-set S of a 2-(v,k,λ) design is the minimum number $i(S)$ of blocks whose union contains S.

Let S denote an s-set in a design D. A block meeting S in exactly i points will be called an *i-secant* block of S. The numbers t_i of i-secant blocks of S are called the *characters* of S. The set S is said to be of *class* $[m_0,m_1,\ldots,m_s]$ with $0\leqslant m_0< m_1<\cdots<m_s\leqslant k$, if we have $t_h=0$ for any $h\notin\{m_0,m_1,\ldots,m_s\}$. If S is of class $[m_0,m_1,\ldots,m_s]$ and, moreover, $t_h\neq 0$ for any $h\in\{m_0,m_1,\ldots,m_s\}$, then S is said to be of *type* $(m_0,m_1,\ldots,m_s)$.

If m and n are, respectively, the minimum and maximum cardinality of $S\cap B$ for all blocks B of D, we say that S is an *$(s;m,n)$-set* of D.

It is easy to prove that the characters of an $(s;m,n)$-set S of D satisfy the following equalities:

$$\begin{cases} \displaystyle\sum_{i=m}^{n} t_i = b, \\ \displaystyle\sum_{i=m}^{n} i t_i = rs, \\ \displaystyle\sum_{i=m}^{n} i(i-1)t_i = \lambda s(s-1). \end{cases} \tag{1}$$

Furthermore, the numbers u_i and v_i of the blocks through a point off S or through a point on S that intersect S in exactly i points, respectively satisfy the following equalities:

$$\begin{cases} \displaystyle\sum_{i=m}^{n} u_i = r, \\ \displaystyle\sum_{i=m}^{n} i u_i = \lambda s, \end{cases} \tag{2}$$

and

$$\begin{cases} \displaystyle\sum_{i=m}^{n} v_i = r, \\ \displaystyle\sum_{i=m}^{n} (i-1) v_i = \lambda(s-1). \end{cases} \tag{3}$$

We recall that in a 2-(v,k,λ) symmetric design of square order there may exist a *Baer subdesign*, that is an s-set intersected by any block in one or n points (i.e., of type $(1,n)$) with $s=[k+(k-\lambda)^{1/2}]/\lambda$ and $n=1+(k-\lambda)^{1/2}$.

In the study of blocking sets in a symmetric design a Baer subdesign plays an important role. We have the following:

Result 1.1 (see [15,16]). *If S is a blocking s-set of a symmetric design D, then*

$$s\geqslant\frac{k+\sqrt{k-\lambda}}{\lambda}.$$

Moreover, equality holds if and only if S is a Baer subdesign of D.

Drake proved the following bound for the cardinality of a blocking set in a design.

Result 1.2 (see [16]). *Let S be a blocking set of a 2-(v,k,λ) design D. Then $k\neq 3$ and*

$$|S|\geqslant\frac{v}{2}-\frac{1}{2k}\sqrt{k^2v^2-4kv^2+4kv}.$$

There are quite a few papers on blocking sets in designs (see for instance [2–8, 12–19]).

A very interesting problem in the theory of blocking sets is to determine the minimum cardinality of a blocking set in a given incidence structure. Moreover, it is interesting to know which designs contain a blocking set having a fixed index.

It is well known that in a projective or affine plane a blocking set has index at least 3. In a previous paper we proved that in a Steiner system $S(t,k,v)$ there is no blocking set of index 1 (cf. [8]). Moreover, it has been proved that any blocking set of an $S(2,k,v)$ has index at least 3 (cf. [18]).

Furthermore, the only biplane containing a blocking s-set of type $(1,s)$ and index 1 is the 2-(4,3,2) design, while each of the five 2-(15,7,3) Hadamard designs contains a blocking 3-set of type $(1,3)$ (cf. [4,6]). In [3] we investigated the existence of a blocking 3-set in a 2-(v,k,λ) design D. This cardinality is the minimum possible if D is different from a 2-$(\lambda+2,\lambda+1,\lambda)$ design (cf. [3]). We established which designs, with $r\geqslant 2\lambda$, may contain a blocking 3-set. These blocking 3-sets have alway index 1.

In Section 2 we give a bound for the cardinality of an intersection $(s;1,n)$-set of a design. We find again, in a different way, Drake's bound for the cardinality of a blocking set of a design; moreover, we characterize the case of equality. Finally, we prove that a 2-(v,k,λ) design with $v>(k-1)^2$ cannot contain a blocking set of index 1.

In Section 3 we deal with the blocking s-sets of type $(1,s)$. We prove some non-existence theorems. Moreover, we establish some bounds for the parameters of a design containing a blocking s-set of type $(1,s)$.

2. Bound for the cardinality of a blocking set

We begin with the following:

Theorem 2.1. *In a* 2-(v,k,λ) *design* D *there is no blocking* s*-set* S *of type* (m).

Proof. First, suppose that in D there is a blocking s-set S of type (1). Then it follows that $s=1$, and $b=r$ in view of (1), a contradiction.

Now, suppose that S is a blocking s-set of type (m) with $m>1$. From (1) we have that $t_m=b$, $mt_m=rs$, $m(m-1)t_m=\lambda s(s-1)$, from which it follows that $mb=rs$, and $(m-1)rs=\lambda s(s-1)$. Consequently, it results that $s=v$ and $m=k$, a contradiction, since S is a blocking set. □

It follows that

Corollary 2.2. *If* S *is a blocking* s*-set of a* 2-(v,k,λ) *design, then* S *is an* $(s;m,n)$*-set with* $1\leqslant m<n$.

We deal with irreducible blocking sets, so we have that $m=1$ and $t_1\geqslant s$. We prove the following bound for the cardinality of an intersection s-set of a design.

Theorem 2.3. *Let* S *be an intersection* $(s;1,n)$*-set of a* 2-(v,k,λ) *design. Then*

$$\frac{nr+\lambda-\sqrt{(nr+\lambda)^2-4\lambda nb}}{2\lambda}\leqslant s\leqslant\frac{nr+\lambda+\sqrt{(nr+\lambda)^2-4\lambda nb}}{2\lambda}. \tag{4}$$

Moreover, the equalities hold if and only if S *is of type* $(1,n)$.

Proof. Let S be an intersection $(s;1,n)$-set of a 2-(v,k,λ) design. For a fixed integer $N\geqslant n$, we obtain by (1) that

$$0\leqslant\sum_{i=1}^{n}(N-i)(i-1)t_i=Nrs-Nb-\lambda s^2+\lambda s,$$

so that

$$\lambda s^2-(Nr+\lambda)s+Nb\leqslant 0. \tag{5}$$

For $N=n$ it follows that

$$\lambda s^2-(nr+\lambda)s+nb\leqslant 0, \tag{6}$$

which implies (4).

Since the equality $\sum_{i=1}^{n}(N-i)(i-1)t_i=0$ is verified if and only if $N=n$ and $t_i=0$ for each i with $1<i<n$, it follows that the equalities in (4) hold if and only if S is of type $(1,n)$. □

By Theorem 2.3 we obtain Drake's bound [16] (cf. Theorem 1.2). In fact, we have the following theorem.

Theorem 2.4. *If an $(s;1,n)$-set is a blocking s-set of a 2-(v,k,λ) design D, then it holds that*

$$\frac{v}{2}-\frac{1}{2k}\sqrt{k^2v^2-4kv^2+4kv}\leqslant s\leqslant\frac{v}{2}+\frac{1}{2k}\sqrt{k^2v^2-4kv^2+4kv}.$$

Moreover, the equalities hold if and only if S is of type $(1,k-1)$.

Proof. If an intersection $(s;1,n)$-set S is a blocking s-set, then $n\leqslant k-1$. Putting $N=k-1$ in (5), we obtain that $s^2-vs+v(v-1)/k\leqslant 0$, that is $ks^2-kvs+v(v-1)\leqslant 0$, which is verified if and only if

$$\frac{v}{2}-\frac{1}{2k}\sqrt{k^2v^2-4kv^2+4kv}\leqslant s\leqslant\frac{v}{2}+\frac{1}{2k}\sqrt{k^2v^2-4kv^2+4kv}.$$

Moreover, in view of Theorem 2.3 equality on the left hand or on the right hand side holds if and only if S is of type $(1,k-1)$. □

In the next theorem we deal with blocking sets of index 1.

Theorem 2.5. *Let S be a blocking $(s;1,n)$-set of index 1 of a 2-(v,k,λ) design D. Then*

$$s\geqslant\frac{v-1}{k}+1.$$

Moreover, equality holds if and only if S is of type $(1,s)$.

Proof. If S is a blocking $(s;1,n)$-set of index 1, then S is an intersection $(s;1,s)$-set. By (6) we have that $\lambda s^2-(sr+\lambda)s+sb\leqslant 0$, from which it follows that $(\lambda-r)s-\lambda+b\leqslant 0$. Thus,

$$s\geqslant\frac{b-\lambda}{r-\lambda}=\frac{v-1}{k}+1.$$

So, the assertion is proved. □

Corollary 2.6. *If in a 2-(v,k,λ) design D there exists a blocking s-set S of index 1, then $v\leqslant(k-1)^2$.*

Proof. If S is a blocking s-set of D of index 1, then $s<k$. So, from Theorem 2.5 we have that $(v-1)/k+1\leqslant s\leqslant k-1$, which implies that $v\leqslant(k-1)^2$. □

Corollary 2.7. *For a fixed positive integer k there are only finitely many 2-(v,k,λ) designs having a blocking set of index 1.*

The bound of Corollary 2.6 on v for the existence of a blocking set of index 1 is better than the bound found in [18].

3. Blocking s-sets of type $(1, s)$

In this section we deal with blocking s-sets of type $(1,s)$ in 2-(v,k,λ) designs D. In the sequel we consider 2-(v,k,λ) designs D that are not 2-$(\lambda+2,\lambda+1,\lambda)$ designs. Consequently, a blocking set has at least three points (cf. [3]). Moreover, if a 2-(v,k,λ) design contains a blocking set, then $k>3$ (cf. [16,18]). We begin with a consequence of Theorem 2.5.

Theorem 3.1. *If a* 2-(v,k,λ) *design D contains a blocking s-set S of type* $(1,s)$, *then* $v=(s-1)k+1$.

Proof. The assertion follows from Theorem 2.5. □

Hence, a 2-(v,k,λ) design D containing a blocking s-set of type $(1,s)$ has $v=\rho k+1$, with $\rho=s-1$, and $1\leqslant\rho\leqslant k-2$.

In particular, from Theorem 3.1 we obtain the following well-known result.

Corollary 3.2. *If a symmetric* 2-(v,k,λ) *design D contains a blocking s-set of type* $(1,s)$, *then* $\lambda\geqslant 2$.

Proof. If D is a symmetric 2-(v,k,λ) design, in view of Theorem 3.1 we have $v=k(k-1)/\lambda+1=\rho k+1$, from which it follows that $\rho=(k-1)/\lambda$. Since $\rho\leqslant k-2$, it holds that $\lambda\geqslant(k-1)/(k-2)>1$. □

Now we deal with some particular cases.

Theorem 3.3. *Let D be a* 2-$(\rho k+1,k,\lambda)$ *design with $k=p+1$, p a prime, and* $1\leqslant\rho\leqslant k-2$. *If D contains a blocking s-set S of type* $(1,s)$, *then $\rho=1$ and $\lambda=p$.*

Proof. Since $r=\rho k\lambda/(k-1)$ is an integer and $\gcd(k-1,k)=1$, it follows that $\rho\lambda=\alpha(k-1)$, where $\alpha\geqslant 1$ is an integer. Since $k-1=p$, and $\rho\leqslant k-2$, we have that $\lambda=k-1$. So, $r=\rho(p+1)$.

If P is a point off S, from (2) we see that $(s-1)u_s=\lambda s-r$. Since $s=\rho+1$, $\lambda=p$, and $r=\rho(p+1)$, we obtain that $\rho u_s=p-\rho$. Since u_s is an integer, and $\rho\leqslant p-1$, it follows that $\rho=1$, and D is a 2-$(p+2,p+1,p)$ design. □

Theorem 3.4. *The only* 2-$(k+1,k,\lambda)$ *design D containing a blocking s-set of type* $(1,s)$ *is the complete* 2-$(k+1,k,k-1)$ *design. Moreover* $s=2$.

Proof. Since D has $r = k\lambda/(k-1)$, necessarily $\lambda = k-1$. Clearly, a simple 2-$(k+1, k, k-1)$ design is complete. Moreover, $s=2$ in view of Theorem 2.5. $\square$

Theorem 3.5. *In a* 2-$((k-1)^2, k, \lambda)$ *design* D, *with* $\lambda \leqslant k$, *there is no blocking* s*-set of type* $(1,s)$.

Proof. Suppose that in a 2-$((k-1)^2, k, \lambda)$ design there is a blocking s-set of type $(1,s)$. Then, in view of Theorem 2.5, $s = k-1$. Since $r = k(k-2)\lambda/(k-1)$ is an integer, it follows that $\lambda = k-1$. Moreover, from (2) we have that $u_{k-1} = 1/(k-2)$. Since $k>3$, it follows that u_{k-1} is not an integer, a contradiction. $\square$

Now we consider the 2-$(\rho k+1, k, \lambda)$ designs with $2 \leqslant \rho \leqslant k-3$, and $k \neq p+1$ for any prime p. We prove the following:

Theorem 3.6. *If in a* 2-$(\rho k+1, k, \lambda)$ *design* D *with* $2 \leqslant \rho \leqslant k-3$ *there is a blocking* s*-set* S *of type* $(1,s)$, *then it holds that* $\rho \leqslant \lambda - 1$ *and* $k \leqslant \lambda^2 - \lambda + 1$.

Proof. Since $r = \rho k\lambda/(k-1)$, and $\gcd(k, k-1) = 1$, it follows that $\rho\lambda = \alpha(k-1)$, where $\alpha \geqslant 1$ is an integer. Being $\rho\lambda \leqslant \lambda(k-3)$, it holds that $\alpha \leqslant \lambda(k-3)/(k-1) < \lambda$. The number u_s of blocks through a point off S that intersects S in exactly s points is

$$u_s = \lambda - (\alpha k - \lambda)/\rho = \lambda - (\rho\lambda + \alpha - \lambda)/\rho = (\lambda - \alpha)/\rho.$$

Since $\alpha < \lambda$, it follows that each point of $D - S$ satisfies $u_s \geqslant 1$, which implies $\lambda - \alpha \geqslant \rho$. So, we have that $1 \leqslant \alpha \leqslant \lambda - \rho$, from which we obtain $\rho \leqslant \lambda - 1$.

From the inequalities $k \leqslant r = \alpha k = \rho\lambda + \alpha \leqslant \rho\lambda + \lambda - \rho \leqslant \lambda^2 - \lambda + 1$ it follows that $k \leqslant \lambda^2 - \lambda + 1$. So, the assertion is proved completely. $\square$

From Theorem 3.6 we have the following consequence.

Corollary 3.7. *If in a* 2-$(\rho k+1, k, \lambda)$ *design* D *there exists a blocking* s*-set of type* $(1,s)$, *then the blocks through the* s *points of the blocking set cover* D.

The following theorem is about the symmetric case.

Theorem 3.8. *Let* D *be a* 2-(v,k,λ) *symmetric design different from a* 2-$(\lambda+2, \lambda+1, \lambda)$ *design* D_λ. *Suppose that* D *contains a blocking* s*-set* S *of type* $(1,s)$. *Then* D *is a* 2-$((\lambda^2 - \lambda - \alpha\lambda + 1)(\lambda - \alpha - 1) + 1, (\lambda^2 - \lambda - \alpha\lambda + 1), \lambda)$ *design, where* α *is an integer with* $0 \leqslant \alpha \leqslant \lambda - 3$. *Moreover, if* $\lambda < 5$, *then it holds that* $\alpha = 0$ *and* $s = \lambda$; *if* $\lambda \geqslant 5$, *then we have that either* $\alpha = 0$ *or* $\alpha = j(\lambda-1)/(j+1)$, *where* j *is an integer with* $1 \leqslant j \leqslant (\lambda-3)/2$, *such that* $j+1$ *divides* $\lambda - 1$. *In this case we have that* $s = \lambda - \alpha$.

Proof. In view of Theorem 3.6, D is a 2-$(\rho k+1,k,\lambda)$ design, with $2\leqslant\rho\leqslant\lambda-1$ and $k\leqslant\lambda^2-\lambda+1$. Put $k=\lambda^2-\lambda+1-\beta$, where β is an integer with $0\leqslant\beta\leqslant\lambda^2-2\lambda$.

If D is a symmetric design, Theorem 2.5 implies that $s=\lambda-\beta/\lambda$. Then $\beta=\alpha\lambda$, where $\alpha\geqslant 0$ is an integer.

Moreover, it holds that $s\geqslant 3$, since D is different from D_λ (cf. [3]). Consequently, $\alpha\leqslant\lambda-3$, so that $0\leqslant\alpha\leqslant\lambda-3$.

If P is a point off S, then (2) implies that $u_s=u_{\lambda-\alpha}=(\lambda-1)/(\lambda-\alpha-1)=1+\alpha/(\lambda-1-\alpha)$. Since $u_{\lambda-\alpha}$ is an integer, we have that either $\alpha=0$ or $\alpha=j(\lambda-1)/(j+1)$, where j is an integer with $1\leqslant j\leqslant(\lambda-3)/2$, which implies that $\lambda\geqslant 5$. Since $\gcd(j,j+1)=1$, $j+1$ divides $\lambda-1$.

Finally, the equalities $s=\lambda$ or $s=\lambda-\alpha$ follow by Theorem 2.5. □

The previous theorem implies the following consequence.

Corollary 3.9. *Let D be a 2-(v,k,λ) symmetric design with $\lambda=p+1$, p a prime. If D contains a blocking s-set of type $(1,s)$, then D is a 2-$(p^3+p^2+p+1,p^2+p+1,p+1)$ design and $s=p+1$.*

Proof. Use the same notation as in the proof of Theorem 3.8. The hypothesis $\lambda=p+1$ implies that the unique possible case is $\alpha=0$. So, the assertion follows. □

We conclude with the following remarks.

Remark 3.10. The design $PG_2(3,q)$ is a 2-$((\lambda-1)(\lambda^2-\lambda+1)+1,\lambda^2-\lambda+1,\lambda)$ symmetric design with $\lambda=q+1$, $q=p^h$, p a prime. In $PG_2(3,q)$ the only blocking $(q+1)$-set of type $(1,q+1)$ is a line of $PG_2(3,q)$ (cf. [19]).

Remark 3.11. A 2-$((\lambda^2-\lambda-\alpha\lambda+1)(\lambda-\alpha-1)+1,(\lambda^2-\lambda-\alpha\lambda+1),\lambda)$ design D is a Hadamard design if either $\alpha=0$ and $\lambda=3$, or $\alpha=\lambda-3$. If $\alpha=0$, D is a 2-$(15,7,3)$ design. We proved that each of the five 2-$(15,7,3)$ Hadamard designs contains a blocking 3-set. Moreover, a blocking 3-set in a 2-$(15,7,3)$ Hadamard design is a Baer subdesign (cf. [6]).

Remark 3.12. In [3] we investigated the problem of the existence of a blocking 3-set in a design. We proved that if a blocking 3-set exists in a 2-(v,k,λ) design D, with $r\geqslant 2\lambda$, then D is one of the following designs: a 2-$(2\lambda+3,\lambda+1,\lambda)$, a 2-$(2(\lambda+1),\lambda+1,\lambda)$, a 2-$(2\lambda-1,\lambda,\lambda)$, a 2-$(4\lambda+3,2\lambda+1,\lambda)$ Hadamard design with λ odd, or a 2-$(4\lambda-1,2\lambda,\lambda)$ Hadamard design. In any case, a blocking 3-set has index one.

References

[1] E.F. Assmus, J.D. Key, Designs and their Codes (Cambridge University Press, Cambridge, 1992).

[2] L. Berardi, Irreducible blocking sets in a symmetric design, Boll. Un. Mat. Ital. A 1 (1987) 39–47.

[3] L. Berardi, A note on 3-blocked designs, J. Combin. Des. (to appear).
[4] L. Berardi, A. Beutelspacher, Blocking sets of the known biplanes, Rend. Mat. Appl. (7) 1 (1987) 36–49.
[5] L. Berardi, A. Beutelspacher, Constructing the biplanes of order four from six points, Boll. Un. Mat. Ital. A 2 (1988) 285–289.
[6] L. Berardi, A. Bichara, Blocking sets in symmetric 2-(15, 7, 3) designs, in: Research and Lecture Notes in Mathematics, Proceedings Conference on Combinatorics 88 (Mediterranean Press, 1988) 135–158.
[7] L. Berardi, M. Buratti, S. Innamorati, 4-Blocked Hadamard 3-designs, in: Proceedings Conference on Combinatorics 94 (to appear).
[8] L. Berardi, F. Eugeni, O. Ferri, Sui blocking sets nei sistemi di Steiner, Boll. Un. Mat. Ital. 3 (1984) 141–164.
[9] T. Beth, D. Jungnickel, H. Lenz, Design Theory (B.I. Wissenschaftsverlag, Mannheim, 1985).
[10] A. Beutelspacher, Einfuhrung in die endliche Geometrie I, II, Bibliographisches Institut, Mannheim (1982).
[11] V.N. Bhat, S.S. Shrikhande, Non-isomorphic solutions of some balanced incomplete block designs I, J. Combin. Theory 9 (1970) 174–191.
[12] R.C. Bose, S.S. Shrikhande, Baer subdesigns of symmetric balanced incomplete block design, in: S. Ikeda et al., eds., Essay in Probability and Statistics (Shinka Tsusho Co., Tokyo, 1976) 1–16.
[13] A.A. Bruen, J.A. Thas, Blocking sets, Geom. Dedicata 6 (1977) 193–203.
[14] P.J. Cameron, Biplanes, Math. Z. 131 (1973) 85–101.
[15] M.J. de Resmini, On blocking sets in symmetric BIBD's with $\lambda \geqslant 2$, J. Geom. 18 (1982) 194–198.
[16] D.A. Drake, Blocking sets in block design, J. Combin. Theory Ser. A 2 (1985) 459–462.
[17] F. Eugeni, S. Innamorati, Arcs and blocking sets in symmetric designs, Ars Combin. 24 (1987) 29–45.
[18] F. Eugeni, E. Mayer, Blocking sets of index two, in: Annals of Discrete Mathematics 37 (North-Holland, Amsterdam, 1988) 169–176.
[19] G. Tallini, Blocking sets nei sistemi di Steiner e d-blocking sets in $PG(r, q)$ e in $AG(r, q)$, Sem. Geom. Combin. Fac. Ing. Univ. L'Aquila 3 (1982).